4 More computations of THH

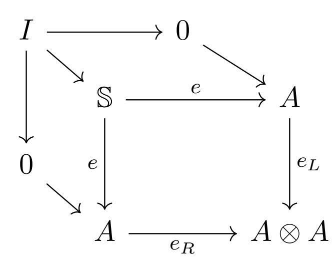
Talk given by Jackson Morris
Notes by Atticus Wang

Fix an odd prime p throughout. In this talk we'll review some background on suspension operations and filtered spectra, and then use some of this technology to compute the homotopy groups of $\mathrm{THH}(\mathbb{Z}_p)/p$ and $\mathrm{THH}(\ell_p)/(p, v_1)$.

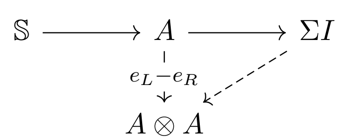
4.1 Suspension operations

Let us first elaborate on the suspension operations mentioned near the end of the last talk. This will be useful for later talks. The main references for this section are [2] appendix A and [3] section 3.1.

Without aiming for full generality, let's work in the familiar land of spectra. Let A be a homotopy commutative $\mathbb{E}_1$-algebra, and let $I = \mathrm{fib}(\mathbb{S} \to A)$. There are two maps $A \to A \otimes_{\mathbb{S}} A$, given by the left and right units. We can precompose both maps with $I \to \mathbb{S} \to A$. Then any point in $I \to A \otimes_{\mathbb{S}} A$ is nullhomotopic "for two reasons", so we obtain a loop in $A \otimes_{\mathbb{S}} A$. In other words we get a map $\Sigma I \to A \otimes A$. This is how you could think about it. Formally, there are natural homotopies filling all the squares

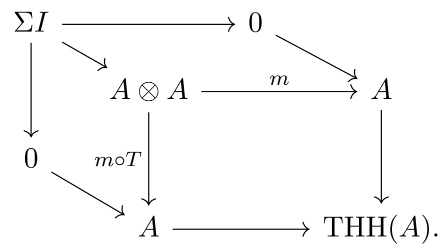


so you automatically get a map $\Sigma I \to A \otimes A$ by pushing out from the back face. Another way to say the same thing is that there is a natural nullhomotopy of the composition $\mathbb{S} \to A \xrightarrow{e_L - e_R} A \otimes A$, so we get a dotted map



This is the suspension map σ.

Doing this once more gives a map $\sigma^2 \colon \Sigma^2 I \to \mathrm{THH}(A)$. In the intuitive picture, the loop you got before in $A \otimes A$ is zero in $\mathrm{THH}(A)$ again for two reasons, so we get a 2-sphere in THH(A). Formally:



Here, we need the natural map $e_L - e_R : A \to \mathrm{fib}(m\colon A \otimes A \to A)$ to factor through ΣI. It suffices to show that $\mathbb{S} \to A \to \mathrm{fib}(m)$ is zero, i.e. that the map $\pi_0(A) \to \pi_0(\mathrm{fib}(m))$ maps the unit to 0. But because m is surjective on homotopy groups, $\pi_0(\mathrm{fib}(m)) \subset \pi_0(A \otimes A)$, so the map $\pi_0(A) \to \pi_0(A \otimes A)$ induced by $e_L - e_R$ does indeed map the unit to zero.

Exercise 4.1. Show that the Bott element $u \in \mathrm{THH}_2(\mathbb{F}_p)$ arising from the identification $\mathrm{THH}(\mathbb{F}_p) \simeq \mathbb{F}_p \otimes \Sigma^\infty_+ \Omega S^3$ is the same element as $\sigma^2 p$. (Hint: first show that σp is the degree 1 exterior power generator of the Steenrod algebra by comparing with $\mathbb{F}_p \otimes^L_{\mathbb{Z}} \mathbb{F}_p$.)

Precomposing σ^2 with the connecting homomorphism gives the map

$$d\colon \Sigma A \to \Sigma^2 I \xrightarrow{\sigma^2} \mathrm{THH}(A).$$

The geometric picture is that this is supposed to take a point in A, take its image in THH(A), and use the S^1 action on THH(A) to form a circle. In other words:

Exercise 4.2. The map d is the same map as

$$\Sigma A \to A \otimes \Sigma A = (\Sigma^\infty_+ S^1) \otimes A \to \mathrm{THH}(A)$$

where the last map is induced by the circle action on THH.

The last map we'll define is a map $t\colon \Sigma^{-2}\,\mathrm{THH}(A) \to \lim_{\mathbb{CP}^1} \mathrm{THH}(A)$, where the limit is taken over the diagram $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^\infty = BS^1 \to \mathrm{Sp}$ encoding the S^1-action on THH. To construct this, write $\mathbb{CP}^1$ as a colimit of spaces

$$\mathbb{CP}^1 = \mathrm{colim}(S^1 \rightrightarrows S^0).$$

When taking the limit over an indexing category which can be written as a colimit, one can break it up as a limit of the limits over each individual part[1]. Applying this to the above cofiber sequence we get

$$\begin{aligned}\lim_{\mathbb{CP}^1} \mathrm{THH}(A) &\simeq \lim(\mathrm{THH}(A) \oplus \mathrm{THH}(A) \rightrightarrows \mathrm{THH}(A) \oplus \Sigma^{-1}\,\mathrm{THH}(A)) \\ &\simeq \lim(\mathrm{THH}(A) \to \Sigma^{-1}\,\mathrm{THH}(A))\end{aligned}$$

and t is just the connecting homomorphism for this fiber sequence.

The geometric picture is that if we take a 2-sphere in THH(A) and apply t, it recovers the fixed points of the S^1-action on the sphere, namely the two poles; in other words it is a kind of inverse to the suspension operation. The geometric picture suggests the following proposition (whose proof we skip):

[1] This is explained e.g. here.

Proposition 4.3 ([2], lemma A.4.1)**.** *The following square commutes:*

$$\begin{array}{ccc} I & \longrightarrow & \mathbb{S} \\ {\scriptstyle \sigma^2}\downarrow & & \downarrow \\ \Sigma^{-2}\,\mathrm{THH}(A) & \xrightarrow[t]{} & \lim_{\mathbb{CP}^1} \mathrm{THH}(A). \end{array}$$

4.2 Filtered objects

In this section we discuss filtered and graded objects. References include [1] appendix B and [3] section 2.

Definition 4.4. Let $\mathcal{C}$ be a stable ∞-category. A *t-structure* on $\mathcal{C}$ is the data of a pair of two subcategories $\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq -1}$ such that:

- $\mathcal{C}_{\geq 0}$ is closed under Σ, and $\mathcal{C}_{\leq -1}$ is closed under Σ^{-1}.
- For $X \in \mathcal{C}_{\geq 0}$ and $Y \in \mathcal{C}_{\leq -1}$, $\mathrm{Map}(X, Y)$ is contractible.
- For any $X \in \mathcal{C}$ there exist $Y \in \mathcal{C}_{\geq 0}$ and $Z \in \mathcal{C}_{\leq -1}$ such that $Y \to X \to Z$ is a fiber sequence.

Exercise 4.5. Show that:

(a) $\mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq -1} = \varnothing$.

(b) There exists a right adjoint $\tau_{\geq 0} : \mathcal{C} \to \mathcal{C}_{\geq 0}$ to the inclusion $\mathcal{C}_{\geq 0} \hookrightarrow \mathcal{C}$. Similarly there exist a left adjoint $\tau_{\leq -1}$.

(c) $\mathcal{C}^{\heartsuit} := \mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq 0}$ is an abelian 1-category.

(d) Given a stable ∞-category $\mathcal{C}$, consider its homotopy category $h\mathcal{C}$ which is a triangulated category. It is clear that a t-structure on $\mathcal{C}$ gives rise to a t-structure on $h\mathcal{C}$. Show that this is a bijection.

Our filtrations are decreasing by convention:

Definition 4.6. Define the category of filtered objects $\mathrm{fil}(\mathcal{C}) = \mathrm{Fun}(\mathbb{Z}_{\geq}, \mathcal{C})$, and the category of graded objects $\mathrm{gr}(\mathcal{C}) = \mathrm{Fun}(\mathbb{Z}, \mathcal{C})$. Both are symmetric monoidal categories by Day convolution. Here $\mathbb{Z}_{\geq}$ is the poset $\cdots 2 \to 1 \to 0 \to -1 \cdots$ and $\mathbb{Z}$ is discrete.

Now suppose $\mathcal{C}$ is a presentably symmetric monoidal stable ∞-category with a t-structure which is compatible with the monoidal structure: this means that the tensor unit is connective (i.e. in $\mathcal{C}_{\geq 0}$), and $\mathcal{C}_{\geq 0}$ is closed under tensor product. The example we'll use is of course Sp. Between the categories $\mathcal{C}, \mathrm{fil}(\mathcal{C}), \mathrm{gr}(\mathcal{C})$, there are the following functors:

- $-_n\colon \mathrm{fil}(\mathcal{C}) \to \mathcal{C}$, taking the nth object.
- the left adjoint to the above, $c^{0,n}\colon \mathcal{C} \to \mathrm{fil}(\mathcal{C})$. This takes X to the filtered object $\cdots \to 0 \to X \to X \to \cdots$ with the first X at the nth position. This is symmetric monoidal. Let $c^{k,n} := \Sigma^k c^{0,n+k}$.

- the colimit (or "realization") colim: $\mathrm{fil}(\mathcal{C}) \to \mathcal{C}$. This is symmetric monoidal.
- the right adjoint to the above, the constant functor $Y\colon \mathcal{C} \to \mathrm{fil}(\mathcal{C})$. This is symmetric monoidal.
- the right adjoint to the above lim: $\mathrm{fil}(\mathcal{C}) \to \mathcal{C}$.
- the associated graded, gr: $\mathrm{fil}(\mathcal{C}) \to \mathrm{gr}(\mathcal{C})$. This is symmetric monoidal.
- the right adjoint to the above, taking zeros for all the filtration maps: $\mathrm{gr}(\mathcal{C}) \to \mathrm{fil}(\mathcal{C})$.
- the Whitehead tower $\tau_{\geq *}\colon \mathcal{C} \to \mathrm{fil}(\mathcal{C})$, which is lax symmetric monoidal.
- the bigraded homotopy groups $\pi^{\heartsuit}_{k,n} X = \pi^{\heartsuit}_k X_{n+k} = \Sigma^{-k}\tau_{\geq k}\tau_{\leq k} X_{n+k} \in \mathcal{C}^{\heartsuit}$ and $\pi_{k,n} X = \pi_k X_{n+k} = [\mathbb{1}^{k,n}, X]$.

Exercise 4.7. Let the *filtration parameter* $\tau \in \pi_{0,-1}(\mathbb{1}^{0,0})$ be defined by the map

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{1} & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{1} & \longrightarrow & \mathbb{1} & \longrightarrow & \cdots \end{array}$$

and let $C\tau \in \mathrm{fil}(\mathcal{C})$.

(a) For any $X \in \mathrm{fil}(\mathcal{C})$, what is $X \otimes \tau$?

(b) Show that $\mathrm{gr}(\mathcal{C})$ can be identified with $\mathrm{Mod}_{C\tau}(\mathrm{fil}(\mathcal{C}))$, and taking associated graded amounts to base changing $- \otimes C\tau$.

(c) On the other hand show that $\mathrm{fil}(\mathcal{C})[\tau^{-1}] \simeq \mathcal{C}$ by $X \mapsto \operatorname{colim} X$.

Proposition 4.8. *Let $X \in \mathrm{fil}(\mathcal{C})$. There is a spectral sequence associated to X:*

$$E^1_{s,t} = \pi^{\heartsuit}_{t-s,s}(\mathrm{gr}\, X) \implies \pi^{\heartsuit}_{t-s}(\operatorname{colim} X).$$

The differentials d_r are of bidegree $(r+1, r)$. Sanity check: d_1 goes from $\pi_{t-s}(\mathrm{gr}_t\, X) \to \pi_{t-s-1} X_{t+1} \to \pi_{t-s-1}(\mathrm{gr}_{t+1}\, X)$ which is correct.

Remark 4.9. Many people prefer for this spectral sequence to be graded according to the Adams convention, where the x-axis represents total degree $t-s$ and the y-axis represents Adams filtration degree s.

We now specialize to $\mathcal{C} = \mathrm{Sp}$ where $\pi^{\heartsuit}$ and π agree. Given a spectrum, we often study it by putting a filtration on it and studying its associated spectral sequence. The suspension operations allow us to formally determine some differentials in the spectral sequence. To start with, given any spectrum X and an element $x \in \pi_*(X)$ with a lift $\widetilde{x} \in \pi_*(X \otimes I)$, we can tensor the maps σ, σ^2 with X to obtain

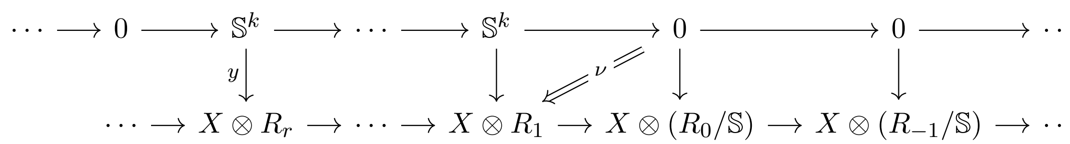
$$\sigma x \in \pi_{*+1}(X \otimes A \otimes A),\ \sigma^2 x \in \pi_{*+2}(X \otimes \mathrm{THH}(A)).$$

In general they depend on the lift $\widetilde{x}$. Similarly given any $y \in \pi_*(X \otimes A \otimes A)$ with a lift $\widetilde{y} \in \pi_*(X \otimes \mathrm{fib}(m))$, we can define $\sigma y \in \pi_{*+1}(X \otimes \mathrm{THH}(A))$. Then for any element $x \in \pi_*(X \otimes A)$ we can define $dx = \sigma((e_L - e_R)x)$.

Now let R be an $\mathbb{E}_1$-algebra in fil(Sp), $X \in \mathrm{Sp}$. Suppose $x \in \pi_k(X)$ whose image in $\pi_k(X \otimes R_0) = \pi_{k,-k}(X \otimes R)$ is equal to $\tau^r y$ for some $y \in \pi_k(X \otimes R_r)$, $r \geq 1$. This means that x maps to zero in $\pi_{k,-k}(X \otimes \mathrm{gr}(R))$. The following is the key fact we'll use to determine the differentials:

Proposition 4.10 ([3], lemma 3.6)**.** *There is a lift $\widetilde{x} \in \pi_{k,-k}(X \otimes \mathrm{fib}(\mathbb{S}^{0,0} \to \mathrm{gr}\, R))$ such that in the spectral sequence associated to $X \otimes \mathrm{THH}(R)$, $\sigma^2 x \in \pi_{k+2,-k-2}(X \otimes \mathrm{THH}(\mathrm{gr}\, R))$ survives to the E_r-page, and there is a differential $d_r(\sigma^2 x) = \pm dy$.*

Sketch of proof. The idea is to map into this spectral sequence from a simpler one where we know the differentials. Namely, a nullhomotopy of x in the associated graded is the same as a homotopy $x \sim \tau^r y$, which is the same as a nullhomotopy ν of $\tau^r y$ in $X \otimes \mathrm{cofib}(\mathbb{S}^{0,0} \to R)$. Then we get a map of filtered objects



we know the d_r differentials of the spectral sequence of the top object because everything has to be killed. That differential maps to a nontrivial differential in the bottom object, which in turn maps to a nontrivial differential in $X \otimes \mathrm{THH}(R)$ by applying σ^2. ☐

4.3 Main calculation

We will now do the calculation of $\mathrm{THH}(\mathbb{Z}_p)/p$ and $\mathrm{THH}(\ell_p)/(p, v_1)$ for p an odd prime, following the streamlined method of [3], example 4.2 and 4.3. Note that this computation goes back to Bökstedt '86 (for $\mathbb{Z}_p$) and McClure–Staffeldt '93 (for ℓ_p).

Our strategy is to put filtrations on $\mathbb{Z}_p$ and ℓ_p, which induce filtrations on their THH and quotients. Moreover, gr commutes with THH (because of monoidality) and quotients (because of exactness), so the associated graded are just THH of $\mathrm{gr}\, \mathbb{Z}_p$ and $\mathrm{gr}\, \ell_p$, which are easy to compute.

Recall that ℓ_p is the connective Adams summand: it is a p-complete spectrum which can be defined as the connective cover of $\mathrm{KU}_{(p)}^{h\mu_{p-1}}$ where μ_{p-1} acts via Adams operations. It satisfies

$$\mathrm{ku}_{(p)} = \ell_p \oplus \Sigma^2 \ell_p \oplus \cdots \oplus \Sigma^{2p-4} \ell_p,$$

and it has homotopy groups $\pi_* \ell_p = \mathbb{Z}_p[v_1]$, $|v_1| = 2p - 2$.

Recall also that $\mathbb{Z}_p$ and ℓ_p are the first two examples in a series of spectra called truncated Brown–Peterson spectra, $\mathrm{BP}\langle n \rangle$. They are obtained from BP by quotienting by the elements $v_{n+1}, v_{n+2}, \dots \in \pi_* \mathrm{BP}$. But $\mathbb{Z}_p$ and ℓ_p are highly structured: they are $\mathbb{E}_\infty$-rings, whereas $\mathrm{BP}\langle n \rangle$ in general do not admit such structures.

Let's begin. We filter $\mathbb{Z}_p$ and ℓ_p by:

- $\mathbb{Z}_p^{\mathrm{fil}}$ is given the filtration $\cdots \to \mathbb{Z}_p \xrightarrow{p} \mathbb{Z}_p \xrightarrow{p} \mathbb{Z}_p = \mathbb{Z}_p = \dots$ and let $\widetilde{v_0}$ be $1 \in \mathbb{Z}_p$ in filtration degree 1. It is an $\mathbb{E}_\infty$-ring in fil(Sp), whose associated graded is $\mathbb{F}_p[v_0]$. (By the way, this is also the Adams filtration of $\mathbb{Z}_p$ along $\mathbb{S} \to \mathbb{F}_p$.)
- ℓ^{fil} is given the Whitehead filtration $\cdots \to \tau_{\geq 2}\ell \to \tau_{\geq 1}\ell \to \ell = \ell = \dots$ and let p be in filtration degree 0 and let $\widetilde{v_1}$ be in filtration degree $2p-2$. It is an $\mathbb{E}_\infty$-ring in fil(Sp) because $\tau_{\geq *}$ is lax symmetric monoidal.

We can take $\mathrm{THH}(\mathbb{Z}_p^{\mathrm{fil}})$ in the symmetric monoidal category fil($\mathcal{C}$). The image of $\widetilde{v_0}$ in $\mathbb{Z}_p^{\mathrm{fil}} \to \mathrm{THH}(\mathbb{Z}_p^{\mathrm{fil}})$ is also denoted $\widetilde{v_0}$. We'll compute $\mathrm{THH}(\mathbb{Z}_p^{\mathrm{fil}})/\widetilde{v_0}$. Note that colimits and taking associated graded both commute with THH and cofibers, so $\operatorname{colim}\mathrm{THH}(\mathbb{Z}_p^{\mathrm{fil}})/\widetilde{v_0} = \mathrm{THH}(\mathbb{Z}_p)/p$ and

$$\mathrm{gr}(\mathrm{THH}(\mathbb{Z}_p^{\mathrm{fil}})/\widetilde{v_0}) = \mathrm{THH}(\mathbb{F}_p[v_0])/v_0 = \mathbb{F}_p[\sigma^2 p] \otimes \wedge[dv_0].$$

Here $\sigma^2 p$ has bidegree $(-2,0)$ and dv_0 has bidegree $(0,1)$. By proposition 4.10 applied to $R = \mathbb{Z}_p^{\mathrm{fil}}$, we get $d_1(\sigma^2 p) = dv_0$ in the spectral sequence for $\mathrm{THH}(\mathbb{Z}_p^{\mathrm{fil}})$. This maps to the spectral sequence for $\mathrm{THH}(\mathbb{Z}_p^{\mathrm{fil}})/\widetilde{v_0}$, so the relation holds for us as well. By Leibniz rule, this determines all differentials. There are no extension problems for degree reasons ($(\sigma t_1)^2 = 0$) and we get

$$\pi_*(\mathrm{THH}(\mathbb{Z}_p)/p) \simeq \mathbb{F}_p[\sigma^2 v_1] \otimes \wedge[\sigma t_1]$$

with $|\sigma^2 v_1| = 2p$ and $|\sigma t_1| = 2p-1$. These elements can be named this way by [2], proposition 6.1.6.

Similarly, let's compute $\mathrm{THH}(\ell)/(p, v_1)$. As above, we compute $\mathrm{THH}(\ell^{\mathrm{fil}})/(p, \widetilde{v_1})$, whose colimit is what we want, and whose associated graded is

$$\mathrm{THH}(\mathrm{gr}(\ell^{\mathrm{fil}}))/(p, v_1) = \mathrm{THH}(\mathbb{Z}_p[v_1])/(p, v_1) = \mathbb{F}_p[\sigma^2 v_1] \otimes \wedge[\lambda_1, dv_1],$$

where $|\sigma^2 v_1| = (-2p, 0)$, $|\lambda_1| = (-2p-1, 0)$, and $|dv_1| = (-1, 2p-2)$. By degree reasons only one page can carry differentials, namely the E^{2p-2}-page; and it does have a differential by proposition 4.10, with $d^{2p-2}(dv_1) = \sigma^2 v_1$. Thus the E^∞ page looks like $\mathbb{F}_p[\mu_1] \otimes \wedge[\lambda_1, \lambda_2]$. Again, there are no extension problems for degree reasons, so we conclude that

$$\pi_*(\mathrm{THH}(\ell)/p, v_1) = \mathbb{F}_p[\mu] \otimes \wedge[\lambda_1, \lambda_2]$$

with $|\mu| = 2p^2$, $|\lambda_1| = 2p-1$, $|\lambda_2| = 2p^2-1$.

References

[1] Robert Burklund, Jeremy Hahn, and Andrew Senger. Galois reconstruction of Artin-Tate $\mathbb{R}$-motivic spectra. *arXiv e-prints*, page arXiv:2010.10325, October 2020.

[2] Jeremy Hahn and Dylan Wilson. Redshift and multiplication for truncated Brown-Peterson spectra. *Ann. of Math. (2)*, 196(3):1277–1351, 2022.

[3] David Jongwon Lee and Ishan Levy. Topological hochschild homology of the image of j. *arXiv preprint arXiv:2307.04248*, 2023.