11 Cyclotomic Frobenius and the Segal conjecture

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Fix a prime p throughout. The goal of this talk is to prove the following theorem:

Theorem 11.1 ([3], 4.3.1)**.** *The Frobenius*

$$\varphi\colon \mathrm{THH}(\mathrm{BP}\langle n\rangle)/(v_0,\dots,v_n) \to \mathrm{THH}(\mathrm{BP}\langle n\rangle)^{tC_p}/(v_0,\dots,v_n)$$

is an isomorphism in π_* for all $*$ sufficiently large.

To explain the context for this theorem, recall that the classical Segal's conjecture for C_p is equivalent to the fact that the Frobenius for $\mathrm{THH}(\mathbb{S}) = \mathbb{S}^{\mathrm{triv}}$,

$$\varphi\colon \mathbb{S} \to \mathbb{S}^{tC_p},$$

exhibits the right hand side as the p-completion of $\mathbb{S}$.

Definition 11.2. For a bounded below, p-complete, p-cyclotomic spectrum X, we say it satisfies Segal's conjecture if the Frobenius $\varphi\colon X \to X^{tC_p}$ is an isomorphism in π_* for all $*$ sufficiently large.

It follows by the thick subcategory theorem in BP-modules that in theorem 11.1 we can replace each v_i by any arbitrarily large power $v_i^{e_i}$, so that $\mathbb{S}/(v_0^{e_0},\dots,v_n^{e_n})$ exists as a type $n+1$ complex. By the thick subcategory theorem in spectra, we conclude that for any type $n+1$ spectrum E, $E\otimes \mathrm{THH}(\mathrm{BP}\langle n\rangle)$ satisfies Segal's conjecture in the above sense.

Remark 11.3. The significance of Segal's conjecture is not only historical, but it is also one of the key inputs for proving the Quillen–Lichtenbaum conjecture for $\mathrm{BP}\langle n\rangle$. We sketch this deduction given the necessary inputs. Suppose that X satisfies Segal's conjecture, then

$$\varphi\colon X^{hS^1} \to (X^{tC_p})^{hS^1} = X^{tS^1}$$

is also an isomorphism in large degrees. Suppose in addition that the canonical map $\mathrm{can}\colon X^{hS^1} \to X^{tS^1}$ vanishes in all sufficiently large degrees, a property which turns out also to be true for $X = E\otimes \mathrm{THH}(\mathrm{BP}\langle n\rangle)$ where E is a type $n+2$ complex ([3], Section 6). Then we conclude that $E\otimes \mathrm{TC}(\mathrm{BP}\langle n\rangle)$ is bounded above for any type $n+2$ complex E. With some more work, one can in fact show that $\pi_*(E\otimes \mathrm{TC}(\mathrm{BP}\langle n\rangle))$ is finite. By results of Mahowald–Rezk [5] this implies that

$$\mathrm{TC}(\mathrm{BP}\langle n\rangle) \to L_{n+1}^f \mathrm{TC}(\mathrm{BP}\langle n\rangle)$$

is an isomorphism in sufficiently large degrees. The Quillen–Lichtenbaum conjecture is the analogous statement with TC replaced by algebraic K-theory.

The proof strategy for theorem 11.1 is the following. We'll filter $\mathrm{BP}\langle n\rangle$ by the $\mathbb{F}_p$-Adams filtration, and after taking care of convergence issues we only need to prove the theorem for the associated graded, which is a graded $\mathbb{E}_2$-polynomial algebra over $\mathbb{F}_p$. Such a case can be further reduced to the case of a spherical polynomial ring in a single variable, where the result follows from Segal's conjecture for $\mathbb{S}$.

11.1 Descent towers

One of the oldest techniques to understand a spectrum X is to approximate it with a well-understood ring spectrum A. For example, suppose $X = \mathbb{S}$, then the Adams (resp. Adams–Novikov) spectral sequence tries to approximates it by $A = \mathbb{F}_p$ (resp. $A = \mathrm{MU}$).

The way this works is by writing down the cosimplicial spectrum

$$X \otimes A \rightrightarrows X \otimes A \otimes A \substack{\rightarrow\\\rightarrow\\\rightarrow} X \otimes A \otimes A \otimes A \dots$$

whose limit we denote by $X^{\wedge}_A$. Then we can filter $X^{\wedge}_A$ by the *descent tower*

$$\mathrm{desc}^{\geq *}_A X := \lim(\tau_{\geq *}(X \otimes A) \rightrightarrows \tau_{\geq *}(X \otimes A \otimes A) \substack{\rightarrow\\\rightarrow\\\rightarrow} \dots).$$

The functor assigning X to $\mathrm{desc}^{\geq *}_A X$ is a lax symmetric monoidal functor $\mathrm{Sp} \to \mathrm{fil(Sp)}$. This means for example that if our input X is an $\mathbb{E}_2$-ring such as $\mathrm{BP}\langle n \rangle$, its descent tower is a $\mathbb{E}_2$-ring in the category of filtered spectra. See talk 4 and references therein for more on filtered spectra.

The associated graded of $\mathrm{desc}^{\geq *}_A X$ is

$$\mathrm{gr}^*_A X = \Sigma^* \lim(H\pi_*(X \otimes A) \rightrightarrows H\pi_*(X \otimes A \otimes A) \substack{\rightarrow\\\rightarrow\\\rightarrow} \dots).$$

Totalization of a cosimplicial abelian group like this is computed by the homology of the associated chain complex (by taking alternating sums of the cosimplicial maps). In other words,

$$\pi_{-\bullet} \mathrm{gr}^*_A X = H^{\bullet+*}(\pi_*(X \otimes A) \to \pi_*(X \otimes A \otimes A) \to \dots)).$$

Now, suppose in addition that the map $\pi_*(A) \to \pi_*(A \otimes A)$ induced by $1 \otimes \mathrm{unit}$ (equivalently, $\mathrm{unit} \otimes 1$) is flat. Then a standard trick on cohomology theories shows that

$$\pi_*(X \otimes A \otimes A) = \pi_*((X \otimes A) \otimes_A (A \otimes A)) = A_*X \otimes_{A_*} A_*A$$

and similarly for the other terms. This means that the chain complex computing $\mathrm{gr}^*_A X$ can be rewritten as

$$A_*X \to A_*X \otimes_{A_*} A_*A \to A_*X \otimes_{A_*} A_*A \otimes_{A_*} A_*A \to \dots$$

which is the bar complex which computes the Ext groups $\mathrm{Ext}_{A_*A}(A_*A, A_*X)$ of comodules over the Hopf algebroid (A_*, A_*A) (see [9], appendix A1).

Putting this all together, we have the following:

Proposition 11.4. *There is a natural spectral sequence*

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{A_*A}(A_*, A_*X) = \pi_{t-s} \mathrm{gr}^t_A X \implies \pi_{t-s} X^{\wedge}_A.$$

When X is additionally a ring spectrum, this spectral sequence is multiplicative.

We say this spectral sequence converges *conditionally* if the limit

$$\lim_*(\mathrm{desc}^{\geq *}_A X) = 0.$$

For example if X and A are connective, this is clearly satisfied.

Remark 11.5. There is a natural map $X \to X_A^\wedge$. When the fiber of the unit map $\mathbb{S} \to A$ is 1-connective, this map is an equivalence. For example MU satisfies this. In other cases, we can sometimes still determine $X_A^\wedge$. Since $X_A^\wedge$ is A-local, the map $X \to X_A^\wedge$ factors through the Bousfield localization $L_A X$. The map $L_A X \to X_A^\wedge$ is an equivalence in many situations (see [2], theorems 6.5–6.7). For example, when $A = \mathbb{F}_p$, this is satisfied and we get the p-completion $X_p^\wedge$.

Let's see what happens in the case of $X = \mathrm{BP}\langle n\rangle$ and $A = \mathbb{F}_p$. Since $\mathrm{BP}\langle n\rangle$ is p-complete, we get the spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}_*}^{s,t}(\mathbb{F}_p, H_*(\mathrm{BP}\langle n\rangle;\mathbb{F}_p)) \implies \pi_{t-s}\mathrm{BP}\langle n\rangle,$$

where $\mathcal{A}^*$ is the mod p Steenrod algebra and $\mathcal{A}_*$ its dual. By Steve Wilson's result ([10], proposition 1.7) we know the $\mathbb{F}_p$-cohomology

$$H^*(\mathrm{BP}\langle n\rangle;\mathbb{F}_p) = \mathcal{A}^*/(Q_0,\dots,Q_n)$$

where $Q_0,\dots,Q_n$ are defined as in Milnor [7]. Taking its dual we get

$$H_*(\mathrm{BP}\langle n\rangle;\mathbb{F}_p) = \mathcal{A}_*\square_{\wedge(\tau_0,\dots,\tau_n)}\mathbb{F}_p = \mathrm{eq}(\mathcal{A}_* \rightrightarrows \mathcal{A}_*\otimes\wedge(\tau_0,\dots,\tau_n))$$

where the two maps are inclusion into the first factor and the diagonal $\mathcal{A}_* \to \mathcal{A}_*\otimes\mathcal{A}_*$ composed with the quotient map by ξ's and $\tau_{n+1},\tau_{n+2},\dots$. The fact that $H_*(\mathrm{BP}\langle n\rangle;\mathbb{F}_p)$ is extended means that we can apply the change-of-rings theorem ([9], A1.3.13) to get

$$\mathrm{Ext}_{\mathcal{A}_*}^{s,t}(\mathbb{F}_p, H_*(\mathrm{BP}\langle n\rangle;\mathbb{F}_p)) = \mathrm{Ext}_{\wedge(\tau_0,\dots,\tau_n)}^{s,t}(\mathbb{F}_p,\mathbb{F}_p) = \mathbb{F}_p[v_0,\dots,v_n]$$

with the last step by Koszul duality. These classes v_i live in degree $(s,t) = (1, 2p^i-1)$ and detect the classes $p, v_1,\dots,v_n \in \pi_*(\mathrm{BP}\langle n\rangle) = \mathbb{Z}_p[v_1,\dots,v_n]$ respectively. We can choose lifts $\widetilde{v}_i \in \pi_*\,\mathrm{desc}_{\mathbb{F}_p}^{\geq 2p^i-1}\mathrm{BP}\langle n\rangle$. Note that v_0 can be uniquely lifted and so we denote the lift just by v_0 as well. It in fact exists as an element $v_0 \in \pi_0\,\mathrm{desc}_{\mathbb{F}_p}^{\geq 1}\mathbb{S}$ detecting p.

11.2 Reduction to the associated graded

From now on we'll just write desc for $\mathrm{desc}_{\mathbb{F}_p}^{\geq *}$. Consider the descent tower $\mathrm{desc}\,\mathrm{BP}\langle n\rangle$. As we remarked above, it is an $\mathbb{E}_2$-algebra in fil(Sp), therefore we can apply THH to it in fil(Sp), and the resulting filtered spectrum will have underlying object (i.e. colimit) $\mathrm{THH}(\mathrm{BP}\langle n\rangle)$. Furthermore, there is also a cyclotomic Frobenius map in the filtered context, and it is compatible with taking the colimit: the only caveat is that the Frobenius here is twisted by a shear in filtration degree:

Definition 11.6. For a filtered spectrum $X\colon \mathbb{Z}_\geq \to \mathrm{Sp}$, let L_pX denote the filtered spectrum

$$\dots X_1 \to X_0 \xrightarrow{\mathrm{id}} \dots \xrightarrow{\mathrm{id}} X_0 \to X_{-1}\dots$$

where $(L_pX)_{ip} = \dots = (L_pX)_{ip-p+1} = X_i$. More concisely, it is the left Kan extension of X along $p\colon \mathbb{Z}\to\mathbb{Z}$.

Proposition 11.7 ([1], example A.11). *For any $\mathbb{E}_1$-algebra R in* fil(Sp) *(resp.* gr(Sp)*), there is a natural S^1-equivariant map of filtered (resp. graded) spectra*

$$\varphi : L_p \operatorname{THH}(R) \to \operatorname{THH}(R)^{tC_p}$$

compatible with taking the colimit and taking the associated graded.

Recall in the beginning we wanted to show that the Frobenius

$$\varphi\colon\ \operatorname{THH}(\mathrm{BP}\langle n\rangle)/(v_0, \ldots, v_n) \to \operatorname{THH}(\mathrm{BP}\langle n\rangle)^{tC_p}/(v_0, \ldots, v_n)$$

induces an isomorphism in π_* for $* \gg 0$. We can upgrade this to a map of filtered spectra by proposition 11.7, which recovers the original map after taking colimits:

$$\varphi : L_p \operatorname{THH}(\operatorname{desc} \mathrm{BP}\langle n\rangle)/(v_0, \ldots, \widetilde{v}_n) \to \operatorname{THH}(\operatorname{desc} \mathrm{BP}\langle n\rangle)^{tC_p}/(v_0, \ldots, \widetilde{v}_n) \tag{11.1}$$

so we have broken the problem down to two separate parts:

(a) Show that the filtered objects on both sides have limit 0 (i.e. conditional convergence);

(b) Show that the induced map on the associated graded is an equivalence in large enough degrees.

We'll first deal with (a) and leave (b) for the next section. Recall that because desc is symmetric monoidal, $\operatorname{desc} X$ is a $\operatorname{desc} \mathbb{S}$-module for any spectrum X, in particular there is a map

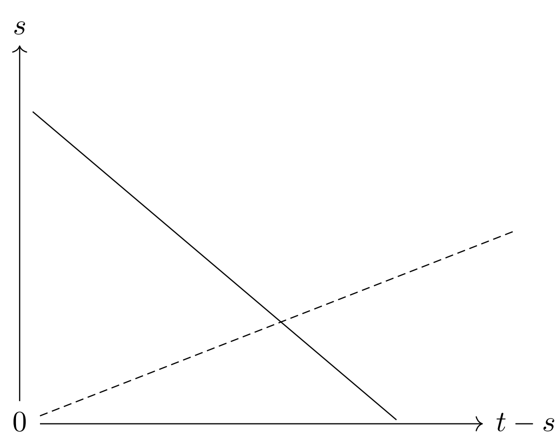
$$v_0 : \operatorname{desc} X[1] \to \operatorname{desc} X$$

where $\operatorname{desc}^{\geq j} X[1] = \operatorname{desc}^{\geq j-1} X$.

Proposition 11.8. *Let A be a connective $\mathbb{E}_1$-algebra, then the tower $\operatorname{THH}(\operatorname{desc} A)^{\wedge}_{v_0}$ converges conditionally.*

Proof. It suffices to show that $\operatorname{THH}(\operatorname{desc} A)/v_0$ converges conditionally. This is computed by the colimit of a simplicial filtered spectrum, with terms $(\operatorname{desc} A)^{\otimes n}/v_0 = \operatorname{desc}(A^{\otimes n})/v_0$.

Let us first show that $\lim(\operatorname{desc}(A^{\otimes n})/v_0) = 0$. In fact it is true for any bounded below spectrum X that $\operatorname{desc}^{\geq *}(X)/v_0$ becomes increasingly connective as $*$ grows. It is computed by a spectral sequence with E_2 page



where the only nonzero entries need to be both above the solid line $t \geq *$ and below the dashed line, which is the classical vanishing line in the Adams spectral sequence (see [3] C.3.4 and references therein). From this it is easy to see that $\operatorname{desc}^{\geq *}(X)/v_0$ has to be increasingly connective.

We now have $U = \mathrm{THH}(\operatorname{desc} A)/v_0$ as a geometric realization of a simplicial filtered spectrum $U(\bullet)$ whose each individual term is conditionally convergent. Consider its skeletal filtration

$$0 \to \mathrm{sk}_0 \to \cdots \to \mathrm{sk}_{n-1} \to \mathrm{sk}_n \to \cdots \to U$$

where sk_n is the realization of the left Kan extension

$$\begin{tikzcd} \Delta^{op}_{\leq n} \arrow[r, "U(\leq n)"] \arrow[d] & \mathrm{fil}(\mathrm{Sp}) \\ \Delta^{op} \arrow[ur, dashed] & \end{tikzcd}$$

Each sk_n is a finite colimit, which is equivalently a finite limit, which commute with limits. So sk_n are all conditionally convergent. The cofiber $\mathrm{sk}_n / \mathrm{sk}_{n-1}$ is the *latching object* $L_{[n]}U$ (see [4]), which is a retract of $\Sigma^n U([n])$. In particular, because the filtered object $U([n]) = \operatorname{desc} A^{\otimes n+1}/v_0$ is uniformly connective in each term, $\mathrm{sk}_n / \mathrm{sk}_{n-1}$ is uniformly n-connective.

Now, fix an index k, and look at Milnor's sequence

$$0 \to \lim\nolimits^1_s \pi_{k-1}(U_s) \to \pi_k(\lim_s U_s) \to \lim_s \pi_k(U_s) \to 0$$

where U_s denotes the filtration on U. For any fixed index s, $\pi_k(U_s) = \pi_k(\operatorname{colim}_n(\mathrm{sk}_n)_s) = \operatorname{colim}_n \pi_k(\mathrm{sk}_n)_s = \pi_k(\mathrm{sk}_{k+1})_s$ by the above. So $\lim_s \pi_k(U_s) = \lim_s \pi_k(\mathrm{sk}_{k+1})_s$. Similarly $\lim^1_s \pi_{k-1}(U_s) = \lim^1_s \pi_{k-1}(\mathrm{sk}_{k+1})_s$. But Milnor's sequence for sk_{k+1} implies that both of these are zero. So we conclude that $\lim U = 0$ as desired. ∎

Proposition 11.9. *Let A be a connective $\mathbb{E}_1$-algebra, then the tower $\mathrm{THH}(\operatorname{desc} A)^{tC_p}$ converges conditionally.*

Proof. We know that $(\lim \mathrm{THH}(\operatorname{desc} A))^{tC_p} = ((\lim \mathrm{THH}(\operatorname{desc} A))^{\wedge}_p)^{tC_p}$ by [8] I.2.9, and the latter is zero because $(\lim \mathrm{THH}(\operatorname{desc} A))/p = \lim(\mathrm{THH}(\operatorname{desc} A)/v_0) = 0$ by proposition 11.8. So it suffices to show that in general for a filtered G-spectrum X which is uniformly bounded below, the natural map $(\lim X)^{tG} \to \lim(X^{tG})$ is an isomorphism.

First, this is clear if we replace tG by hG, so it suffices to prove that the natural map $(\lim X)_{hG} \to \lim(X_{hG})$ is an equivalence. The object X_{hG} is computed by the colimit of the simplicial object

$$\cdots \Sigma^\infty_+ G \otimes \Sigma^\infty_+ G \otimes X \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \Sigma^\infty_+ G \otimes X \rightrightarrows X.$$

Denote its skeleta by $\mathrm{sk}_n(X)$. Then since it is a finite colimit, the natural map $\mathrm{sk}_n(\lim X) \to \lim(\mathrm{sk}_n X)$ is an equivalence. Moreover, since X is uniformly bounded below, the connectivity of the map $\mathrm{sk}_n X \to X_{hG}$ uniformly increases with n. Taking the limit and using Milnor's sequence, we see that the maps $\lim(\mathrm{sk}_n X) \to \lim(X_{hG})$ and $\mathrm{sk}_n(\lim X) \to (\lim X)_{hG}$ both become increasingly connective with n. Thus, the natural map $(\lim X)_{hG} \to \lim(X_{hG})$ is an equivalence. ∎

Combining propositions 11.8 and 11.9 shows (a).

11.3 The Segal conjecture for polynomial $\mathbb{F}_p$-algebras

For part (b), suppose the associated graded object of $\operatorname{desc}\mathrm{BP}\langle n\rangle$ is the $\mathbb{E}_2$-algebra R, then the associated graded of eq. (11.1) is given by

$$\varphi : L_p \,\mathrm{THH}(R)/(v_0, \dots, v_n) \to \mathrm{THH}(R)^{tC_p}/(v_0, \dots, v_n).$$

Recall from the discussion above that $\pi_* R = \mathbb{F}_p[v_0, \dots, v_n]$ as a commutative $\mathbb{F}_p$-algebra, with v_i in total degree $t - s = 2p^i - 2$ and weight[1] $t = 2p^i - 1$. This fact alone determines the $\mathbb{E}_2$-structure of R completely:

Definition 11.10 (spherical polynomial rings)**.** For integers r, w, let $\mathbb{S}^{2r}(w)$ denote the graded spectrum with $\mathbb{S}^{2r}$ in weight w and zero elsewhere. By $\mathbb{S}[x]$ (where x has degree $2r$ and weight w), we will mean the free graded $\mathbb{E}_1$-algebra on $\mathbb{S}^{2r}(w)$, with structural map $x\colon \mathbb{S}^{2r}(w) \to \mathbb{S}[x]$. Recall that as a spectrum it is given by

$$\mathbb{S}^0(0) \oplus \mathbb{S}^{2r}(w) \oplus \mathbb{S}^{4r}(2w) \oplus \dots$$

It in fact admits the structure of a graded $\mathbb{E}_2$-ring ([3], 4.1.1). This $\mathbb{E}_2$ structure originates from the fact that there exists an $\mathbb{E}_2$-map $\mathbb{Z} \to \mathrm{Pic}(\mathbb{S})$ sending $1 \mapsto \mathbb{S}^2$, which comes from the one-point compactification of complex vector spaces (the J-homomorphism).

Proposition 11.11 ([3], proposition 4.2.1)**.** *Let R be a graded $\mathbb{E}_2$-$\mathbb{F}_p$-algebra with $\pi_* R = \mathbb{F}_p[a_1, \dots, a_n]$ with $|a_i|$ in nonnegative even degrees and positive weights. Then R must be equivalent to $\mathbb{F}_p \otimes \mathbb{S}[a_1] \otimes \dots \otimes \mathbb{S}[a_n]$ as graded $\mathbb{E}_2$-$\mathbb{F}_p$-algebras.*

We skip the proof but the idea is to show that $\mathbb{F}_p \otimes \mathbb{S}[a_1] \otimes \dots \otimes \mathbb{S}[a_n]$ has an $\mathbb{E}_2$-$\mathbb{F}_p$-algebras cell structure concentrated in even degrees, which then implies that there are no obsctructions to writing down an $\mathbb{E}_2$ map from it to R, which then must be an equivalence. Granted this, it suffices to prove the following:

Proposition 11.12. *For R as above, the cyclotomic Frobenius*

$$\varphi : L_p \,\mathrm{THH}(R) \to \mathrm{THH}(R)^{tC_p}$$

induces on homotopy groups the obvious inclusion of rings

$$\mathbb{F}_p[x, a_1, \dots, a_n] \otimes \wedge(\sigma a_1, \dots, \sigma a_n) \to \mathbb{F}_p[x^{\pm 1}, a_1, \dots, a_n] \otimes \wedge(\sigma a_1, \dots, \sigma a_n)$$

where x has degree 2 and weight 0, and σ raises degree by 1 and does not change the weight.

This clearly implies (b) since the exterior algebra has bounded degree. See talk 4 for background on the suspension operation σ.

To prove this proposition, we can write φ as the composition

$$\mathrm{THH}(\mathbb{F}_p) \otimes_i L_p \,\mathrm{THH}(\mathbb{S}[a_i]) \to \mathrm{THH}(\mathbb{F}_p)^{tC_p} \otimes_i \mathrm{THH}(\mathbb{S}[a_i])^{tC_p} \to \mathrm{THH}(R)^{tC_p} \tag{11.2}$$

where the first map is the tensor product of each individual Frobenii and the second is from the lax monoidal structure of $(-)^{tC_p}$. Thus, it suffices to show that the first map is of the desired form on homotopy groups, and the second map is an equivalence. To do this, we need to better understand the cyclotomic Frobenius on spherical polynomial rings.

[1] Here, "weight" means the grading from the descent filtration.

11.4 Segal's conjecture for $\mathbb{S}[x]$

Suppose x has degree $2r$ and weight w.

Proposition 11.13 ([6], theorem 3.8)**.** *As a S^1-spectrum,*

$$\mathrm{THH}(\mathbb{S}[x]) = \bigoplus_{k\geq 0} \mathrm{Ind}_{C_k}^{S^1} \mathbb{S}^{2kr}(kw).$$

Here C_k acts on $\mathbb{S}^{2kr}(kw) = (\mathbb{S}^{2r}(w))^{\otimes k}$ by permuting the tensor factors, $\mathrm{Ind}_{C_k}^{S^1}$ is the functor which is left adjoint to the functor $\mathrm{Sp}^{BS^1} \to \mathrm{Sp}^{BC_k}$ restricting along $BC_k \to BS^1$, and the $k = 0$ summand is just $\mathbb{S}$.

As an illustrative example, taking $r = w = 0$ recovers the classical computation

$$\mathrm{THH}(\mathbb{S}[x]) = \mathbb{S}^{\mathrm{triv}} \bigoplus_{k\geq 1} \Sigma^\infty_+(S^1/C_k).$$

Assume from now on that $w > 0$, so that all the different summands of $\mathrm{THH}(\mathbb{S}[x])$ live in distinct weights, and we can look at each k separately.

Definition 11.14. For a topological group G, a G-spectrum X is *finite* if it lies in the thick subcategory of G-spectra generated by $\Sigma^\infty_+(G/H)$ for $H \leq G$ closed.

Proposition 11.15. *If $w \neq 0$, then $\mathrm{THH}(\mathbb{S}[x])$ is weight-wise finite as a C_p-spectrum.*

Proof. Clearly it suffices to look at a fixed k. First, for $r \geq 0$, $(\mathbb{S}^{2r})^{\otimes k}$ is a retract of $\Sigma^\infty_+ S^{2r\rho}$ where ρ is the regular representation of C_k. The space $S^{2r\rho}$ admits a finite C_k-CW-structure, so $\mathbb{S}^{2kr}$ is finite. Now, induction preserves finiteness, because $\mathrm{Ind}_G^F(G/H) = F/H$. So $\mathrm{Ind}_{C_k}^{S^1} \mathbb{S}^{2kr}$ is finite as a S^1-spectrum. We now need to show that any finite S^1-spectrum is finite as a C_p-spectrum. The proper closed subgroups of S^1 are C_n for $n \geq 1$. If $p \mid n$ then C_p fixes S^1/C_n so it's just $\Sigma(\Sigma^\infty_+ *)$. If $p \nmid n$, then C_p acts on S^1/C_n freely, and it also clearly admits a finite C_p-CW-structure. □

Let's now see what $\mathrm{THH}(\mathbb{S}[x])^{tC_p}$ looks like. We can look at each k independently. By the proof above, when $p \nmid k$, $\mathrm{Res}_{C_p}^{S_1} \mathrm{Ind}_{C_k}^{S_1} \mathbb{S}^{2kr}$ can be built only using suspensions of $\Sigma^\infty_+ C_p = \mathbb{S}^{\oplus p}$, whose Tate fixed point is 0 since C_p acts freely. When $k = mp$, we have

$$\mathrm{Res}_{C_p}^{S_1} \mathrm{Ind}_{C_k}^{S_1} \mathbb{S}^{2kr} = \mathbb{S}^{2kr} \otimes \Sigma^\infty_+ S^1$$

where C_p acts on the second factor trivially and on the first factor by permuting the tensor factors in $\mathbb{S}^{2kr} = (\mathbb{S}^{2kr})^{\otimes p}$. Therefore,

$$\mathrm{THH}(\mathbb{S}[x])^{tC_p} \simeq \bigoplus_{m\geq 0} \mathbb{S}^{2rmp}(wmp)^{tC_p} \otimes \Sigma^\infty_+ S^1.$$

Proposition 11.16. *The cyclotomic Frobenius*

$$\varphi : L_p \,\mathrm{THH}(\mathbb{S}[x]) \to \mathrm{THH}(\mathbb{S}[x])^{tC_p}$$

exhibits the right side as the p-completion of the left side.

Proof. By definition of φ we have the following square whose arrows are all S^1-equivariant:

$$\begin{tikzcd} L_p\mathbb{S}[x]^{\otimes m} \otimes \Sigma^\infty_+ S^1 \arrow[r] \arrow[d] & (\mathbb{S}[x]^{\otimes mp})^{tC_p} \otimes \Sigma^\infty_+ S^1 \arrow[d] \\ L_p \operatorname{THH}(\mathbb{S}[x]) \arrow[r, "\varphi"'] & \operatorname{THH}(\mathbb{S}[x])^{tC_p}. \end{tikzcd}$$

Looking at the summand corrsponding to x^m in the top map, it is by definition the Tate-valued Frobenius

$$\mathbb{S}^{2rm} \to (\mathbb{S}^{2rmp})^{tC_p}$$

tensored with S^1_+, which is the p-completion map by the classical Segal's conjecture for spheres. On the bottom, this map corresponds to φ in weight wpm. This proves that φ is p-completion in each weight and thus we are done. $\square$

We can finally prove proposition 11.12, which finishes the proof of the main theorem 11.1.

Proof of proposition 11.12. Recall that we need to show the first map in eq. (11.2) is the desired map on homotopy groups and the second map there is an equivalence. It is a classical fact ([8], IV.4) that the Frobenius $\operatorname{THH}(\mathbb{F}_p) \to \operatorname{THH}(\mathbb{F}_p)^{tC_p}$ is $\mathbb{F}_p[x] \to \mathbb{F}_p[x^{\pm}]$ on homotopy, and we have just shown that $L_p \operatorname{THH}(\mathbb{S}[x]) \to \operatorname{THH}(\mathbb{S}[x])^{tC_p}$ is p-completion, so the first map is indeed as claimed on homotopy groups.

To show the second map is an equivalence, using proposition 11.15, it suffices to show that for X, Y nonnegatively graded C_p-spectra where Y is weight-wise finite, then the natural map $X^{tC_p} \otimes Y^{tC_p} \to (X \otimes Y)^{tC_p}$ is an equivalence. Because they are nonnegatively graded and $(-)^{tC_p}$ is taken degreewise, it suffices to show the same statement for X, Y (ungraded) C_p-spectra. In this case, the subcategory of C_p-spectra Y such that the map is an equivalence is thick and contains $\Sigma^\infty_+ C_p$ (because both sides are 0) and $\mathbb{S}$ (by the classical Segal conjecture), so we are done. $\square$

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